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Determination of Shape Preserving Spline Interpolants with Minimal Curvature via Dual Programs

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1. INTRODUCTION

In this paper programming problems that arise in the convexity or monotonicity preserving spline interpolation will be studied from a numerical point of view. The problems appearing there are of the following special structure.

PROGRAM PA.

$$\begin{aligned} &\text{Minimize} \quad \sum_{i=1}^{n} F_i(m_{i-1}, m_i) \\ &\text{subject to} \quad (m_{i-1}, m_i) \in W_i \qquad (i = 1, ..., n), \end{aligned} \tag{1.1}$$

where $F_i\colon R^2 \to R$ is a strictly convex function and $W_i \subset R^2$ a closed convex set. For solving PA computationally here it is proposed to pass to the dual

PROGRAM DA.

$$\begin{aligned} &\text{Maximize} \quad -\sum_{i=1}^{n} H_i^*(p_{i-1}, -p_i) \\ &\text{with} \qquad p_0 = p_n = 0. \end{aligned} \tag{1.2}$$

The function $H_i^*\colon R^2 \to \bar{R}$ is therein to be defined by

$$H_i^*(\xi, \eta) = \sup\{\xi x + \eta y - F_i(x, y)\colon (x, y) \in W_i\}. \tag{1.3}$$

In DA no constraints occur and, moreover, the gradient of the objective function is tridiagonal. These properties are indeed very convenient for

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numerical purposes and should be utilized by practical algorithms in order to be efficient. Attacking PA directly, e.g., by an active set strategy (see, e.g., Fletcher [2]) is in general more expansive than starting from DA.

Further, if a solution of DA is known then, under suitable assumptions, a solution of PA is explicitly given by the return-formula

$$\begin{aligned} m_{i-1} &= \partial_1 H_i^*(p_{i-1}, -p_i) \\ m_i &= \partial_2 H_i^*(p_{i-1}, -p_i) \qquad (i=1, ..., n). \end{aligned} \tag{1.4}$$

In addition, for problems arising in the shape preserving spline interpolation, the program DA can be explicitly stated because the subprograms (1.3) are solvable by formulas, and strong duality theorems are valid.

Following this line the results of Burmeister *et al.* [1] concerning the convex spline interpolation are confirmed deductively. Moreover, also for the monotone spline interpolation problem, analogous results can be offered now that likewise are obtained in a deductive manner by applying the above-mentioned general concept. For further applications see [11], [13], [14], [15], and [16].

2. Formulation of the Spline Problems

For describing the needed facts from the spline interpolation denote by Δ a partition of the unit interval [0, 1],

$$\Delta: x_0 = 0 < x_1 < \cdots < x_{n-1} < x_n = 1,$$

and let $\mathrm{Sp}(3, \Delta)$ be the set of cubic C^1-splines on Δ. The points $(x_0, y_0), ..., (x_n, y_n)$ associated with Δ are said to be in convex position if

$$\tau_1 \leqslant \tau_2 \leqslant \cdots \leqslant \tau_n, \tag{2.1}$$

where $\tau_i = (y_i - y_{i-1})/h_i$, $h_i = x_i - x_{i-1}$. They are in monotone position if

$$\tau_1 \geqslant 0, ..., \tau_n \geqslant 0. \tag{2.2}$$

Spline interpolants $s \in \mathrm{Sp}(3, \Delta)$ should satisfy the concrete interpolation condition

$$s(x_i) = y_i \qquad (i = 0, ..., n). \tag{2.3}$$

Further, denote by $m_i = s'(x_i)$ the first derivative of s in the node x_i.

Now, a result of Neuman [5] states that under (2.1) a spline interpolant $s \in \mathrm{Sp}(3, \Delta)$ is convex on [0, 1] if and only if

$$(m_{i-1}, m_i) \in W_i \qquad (i = 1, ..., n), \tag{2.4}$$

where $W_i = K_i$ is the cone

$$K_i = \{(x, y) \in R^2 \colon 2x + y \leqslant 3\tau_i,\ x + 2y \geqslant 3\tau_i\}. \tag{2.5}$$

An analogous result for the monotone spline interpolation, under the assumption (2.2), is due to Fritsch and Carlson [3]. Here $W_i = E_i \cup D_i$ is taken as the union of the ellipse

$$E_i = \{(x, y) \in R^2 \colon x^2 + xy + y^2 - 6\tau_i(x + y) + 9\tau_i^2 \leqslant 0\} \tag{2.6}$$

with the square

$$D_i = \{(x, y) \in R^2 \colon 0 \leqslant x \leqslant 3\tau_i,\ 0 \leqslant y \leqslant 3\tau_i\}. \tag{2.7}$$

Naturally for a given set $m_0, ..., m_n$ of first derivatives the spline interpolant $s \in \mathrm{Sp}(3, \Delta)$ is uniquely determined and can be directly computed.

In the convex case the problem (2.4), with $W_i = K_i$, has in general no solution. An effective algorithm by which the solvability can be decided is given by Schmidt and Hess [7]; see also [9], [10], and [12]. If at all, (2.4) has either one solution or an infinite number of solutions. In the monotone case the problem (2.4), with $W_i = E_i \cup D_i$, is always solvable but in general not uniquely. Therefore, both in the convex and in the monotone spline interpolation case the question arises which solution of (2.4), i.e., which of the corresponding splines shall be chosen. A good strategy is to select that spline which has minimal mean curvature

$$\int_0^1 s''(x)^2\,dx = \sum_{i=1}^{n} \frac{4}{h_i} \{m_{i-1}^2 + m_{i-1}m_i + m_i^2 - 3\tau_i(m_{i-1} + m_i) + 3\tau_i^2\}. \tag{2.8}$$

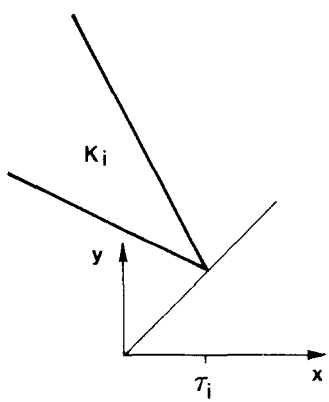

FIG. 1. The cone K_i.

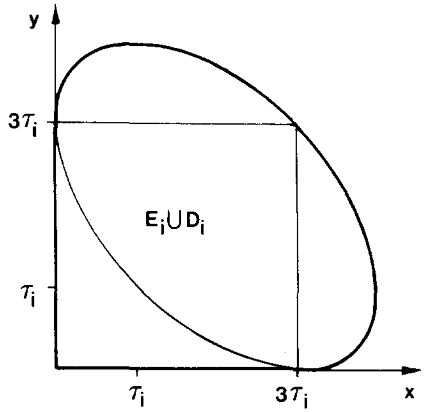

FIG. 2. The set $E_i \cup D_i$.

Thus one is led to a programming problem of the type PA, with the function

$$F_i(x, y) = \frac{4}{h_i} \{x^2 + xy + y^2 - 3\tau_i(x + y) + 3\tau_i^2\} \tag{2.9}$$

and the sets

$$W_i = K_i \qquad \text{or} \qquad W_i = E_i \cup D_i \tag{2.10}$$

introduced by means of (2.5)–(2.7).

The Hessian of the function (2.9) is positive definite. This property implies

PROPOSITION 1. *The program PA and the auxiliary programs* (1.3) *are uniquely solvable if specified by* (2.9) *and* (2.10) *and if the feasible domains are nonempty. The optimal values are finite.*

3. APPLICATIONS OF FENCHEL'S THEORY

In order to verify the duality of the above-mentioned pair PA and DA of programs and to derive duality statements one can refer to Fenchel's theory and to extensions of it due to Rockafellar and Stoer, see, e.g., [4, 6, 8], but also Lagrange's duality theory may be applied.

Let f, g: $R^m \to \bar{R} = R \cup \{+\infty, -\infty\}$ be proper convex functions, that is, f and g are convex functions greater than $-\infty$ whose effective domains $\operatorname{dom} f$ and $\operatorname{dom} g$ are nonempty. Moreover, let f and g be closed. As usual, the subdifferential of f is denoted by ∂f while the conjugate function will be called f^*,

$$f^*(v) = \sup\{u^{\mathrm{T}}v - f(u): u \in R^m\}. \tag{3.1}$$

Further remember that for the indicator function I of a set W is $I(u; W) = 0$ for $u \in W$ and $I(u; W) = +\infty$ for $u \notin W$. The subsequently used concepts of stability are defined, e.g., in [8].

Now, in Fenchel's theory, to the primal convex program

$$P: \inf\{f(u) + g(u): u \in R^m\} \tag{3.2}$$

belongs the dual concave program

$$D: \sup\{-f^*(v) - g^*(-v): v \in R^m\}. \tag{3.3}$$

A corresponding strong duality theorem of a generality necessary here reads as follows (see [8, Section 5.3] for the first part and see, e.g., [6, Section 31] for the second part).

THEOREM 2. *Let f and g be closed proper convex functions. Assume that*

$$\inf\{f(u) + g(u): u \in R^m\} \in R \tag{3.4}$$

and that there are vectors w, $z \in \operatorname{dom} f \cap \operatorname{dom} g$ such that f is w-stable and g is z-stable. Then, program D has an optimal solution, and the optimal values of P and D are equal:

$$\inf\{f(u) + g(u): u \in R^m\} = \max\{-f^*(v) - g^*(-v): v \in R^m\}. \tag{3.5}$$

In addition, u and v are optimal for P and D, respectively, if and only if

$$u \in \partial f^*(v) \cap \partial g^*(-v) \tag{3.6}$$

holds.

Next, the programs PA and DA are shown to be special cases of P and D while the verification of the assumptions of the preceding theorem including the application of formula (3.6) will be undertaken later and only for the concrete spline problems.

By introducing additional variables, Program PA can be reformulated as

PA′: Minimize $\sum_{i=1}^{n} F_i(m_{i-1}, M_i)$

subject to $(m_{i-1}, M_i) \subset W_i$ $(i = 1, \ldots, n)$, $M_i = m_i$ $(i = 1, \ldots, n-1)$,

while $M_n = m_n$ is only a change of notation.

Naturally this is a separable program treated by several authors, see, e.g., [6, Section 31]. Using the function $H_i\colon R^2 \to \bar{R}$,

$$H_i(x, y) = F_i(x, y) + I(x, y; W_i), \tag{3.7}$$

set, with $u = (m_0, M_1, m_1, ..., M_{n-1}, m_{n-1}, M_n) \in R^{2n}$,

$$f(u) = \sum_{i=1}^{n} H_i(m_{i-1}, M_i). \tag{3.8}$$

Further let

$$g(u) = I(u; W), \qquad W = \{u \in R^{2n}\colon M_i = m_i (i = 1, ..., n-1)\}. \tag{3.9}$$

By these definitions problem PA′ and therefore likewise PA become a program P. The occurring functions f and g are proper convex and closed.

In order to state the corresponding dual program D the conjugate functions f^* and g^* to (3.8) and (3.9) are to be determined. Let $v = (p_0, P_1, p_1, ..., P_{n-1}, p_{n-1}, P_n) \in R^{2n}$. Then, in view of (3.8), it follows immediately that

$$f^*(v) = \sum_{i=1}^{n} H_i^*(p_{i-1}, P_i) \tag{3.10}$$

holds if H_i^* is given by (1.3). Observe further that

$$\begin{aligned} g^*(v) &= \sup\{u^{\mathrm{T}} v\colon u \in W\} \\ &= 0 \qquad \text{for} \quad p_0 = 0, P_i + p_i = 0\ (i = 1, ..., n), p_n = 0 \\ &= +\infty \qquad \text{otherwise} \end{aligned} \tag{3.11}$$

and therefore $g^*(-v) = g^*(v)$. Hence, problem D reduces to program DA.

Now, the formula (3.6) shall be specified. To this end, it is additionally assumed that the auxiliary programs (1.3) are solvable for all $(\xi, \eta) \in R^2$ with finite optimal values $H_i^*(\xi, \eta)$. Because of the strict convexity of F_i the maximizer

$$(\bar{x}_i(\xi, \eta), \bar{y}_i(\xi, \eta))$$

in (1.3) is unique. Further, since $(x_i, y_i) \in \partial H_i^*(\xi, \eta)$ if and only if (x_i, y_i) is a maximizer of (1.3), one gets the differentiability of H_i^* and

$$\begin{aligned} \partial_1 H_i^*(\xi, \eta) &= \bar{x}_i(\xi, \eta), \\ \partial_2 H_i^*(\xi, \eta) &= \bar{y}_i(\xi, \eta). \end{aligned} \tag{3.12}$$

In addition, the derivatives $\partial_1 H_i^*$ and $\partial_2 H_i^*$ are continuous. For this, see, e.g., [6, Sections 23 and 25]. Thus, f^* defined by (3.10) is continuously differentiable.

Further, let v be optimal for D and let the optimal value be finite. This implies $g^*(-v)=0$, and the Kuhn–Tucker conditions for program D, i.e., for

$$\inf\{f^*(v): p_0=0,\ P_i+p_i=0\ (i=1, ..., n-1),\ P_n=0\},$$

yield $\partial_1 H_{i+1}^*(p_i, P_{i+1})=\partial_2 H_i^*(p_{i-1}, P_i)$, $i=1, ..., n-1$. Now, define the vector u by

$$m_{i-1}=\partial_1 H_i^*(p_{i-1}, P_i), \qquad M_i=\partial_2 H_i^*(p_{i-1}, P_i), \qquad i=1, ..., n.$$

Then one gets $g(u)=0$, $v^{\mathrm{T}}u=0$, and hence $g^*(-v)+g(u)=-v^{\mathrm{T}}u$. Thus it follows that $u\in\partial g^*(-v)$, see, e.g., [6, Section 23]. Further, because of $u\in\partial f^*(v)$, relation (3.6) is valid. Thus, the mentioned assumptions imply

PROPOSITION 3. *Let* $(p_0, p_1, ..., p_n)$ *be a solution of DA then* $(m_0, m_1, ..., m_n)$ *with*

$$\begin{aligned} m_{i-1}&=\partial_1 H_i^*(p_{i-1}, -p_i)=\bar{x}_i(p_{i-1}, -p_i), \\ m_i&=\partial_2 H_i^*(p_{i-1}, -p_i)=\bar{y}_i(p_{i-1}, -p_i) \qquad (i=1, ..., n) \end{aligned} \tag{3.13}$$

is the only solution of program PA. The derivatives $\partial_1 H_i^*$, $\partial_2 H_i^*$ *are continuous.*

4. CONVEX SPLINE INTERPOLATION

As said above, for determining the convex spline intepolant with minimal mean curvature one has to solve the

PROGRAM PC.

$$\begin{aligned} &\text{Minimize} \quad \sum_{i=1}^{n} F_i(m_{i-1}, m_i) \\ &\text{subject to} \quad (m_{i-1}, m_i)\subset K_i \qquad (i=1, ..., n) \end{aligned} \tag{4.1}$$

with F_i and K_i according to (2.9) and (2.5), respectively. By Proposition 1 this program is uniquely solvable if the feasible domain

$$K=\{(m_0, m_1, ..., m_n)\in R^{n+1}: (m_{i-1}, m_i)\in K_i (i=1, ..., n)\} \tag{4.2}$$

is not empty. Note that $K\neq\varnothing$ implies (2.1).

For numerical purposes the corresponding dual program DC shall be stated. To this end, the subprogram (1.3), now

$$\begin{aligned} &\text{maximize } \mathrm{d}(x, y) = \xi x + \eta y - \frac{4}{h}\{(x-\tau)^2 + (x-\tau)(y-\tau) + (y-\tau)^2\} \\ &\text{subject to } 2x + y \leqslant 3\tau,\ x + 2y \geqslant 3\tau, \end{aligned} \tag{4.3}$$

where $h = h_i$ and $\tau = \tau_i$, is to be treated for all $(\xi, \eta) \in R^2$. In view of (1.2) and (1.4) especially, the optimal value $d_{\max} = H_i^*(\xi, \eta)$ is of interest. Proposition 3 assures that H_i^* is continuous differentiable.

PROPOSITION 4. *The optimal value of program* (4.3) *is equal to*

$$\begin{aligned} H_i^*(\xi, \eta) &= \tau_i(\xi + \eta) + \frac{h_i}{12}(\xi^2 - \xi\eta + \eta^2) && \textit{for } \xi \leqslant 0, \eta \geqslant 0 \\ &= \tau_i(\xi + \eta) + \frac{h_i}{12}\left(\frac{\xi}{2} - \eta\right)^2 && \textit{for } \xi \geqslant 0, \xi \leqslant 2\eta \\ &= \tau_i(\xi + \eta) + \frac{h_i}{12}\left(\xi - \frac{\eta}{2}\right)^2 && \textit{for } \eta \leqslant 0, 2\xi \leqslant \eta \\ &= \tau_i(\xi + \eta) && \textit{for } \xi \geqslant 2\eta, 2\xi \geqslant \eta. \end{aligned} \tag{4.4}$$

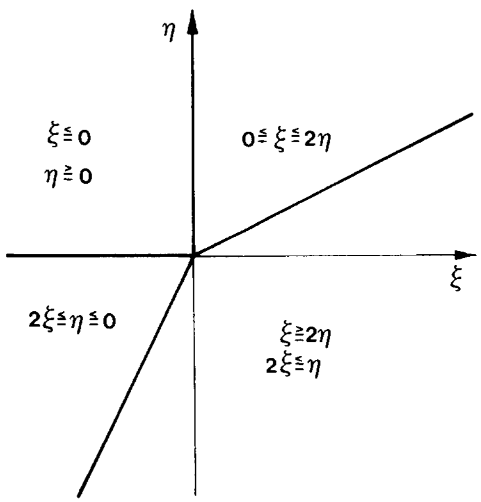

FIG. 3. Pieces of the domain of function (4.4).

The proof of (4.4) is elementary and is therefore only sketched. In the first case, the center of the ellipse $d(x, y)=c$,

$$\tilde{x}=\tau+\frac{h}{12}(2\xi-\eta), \qquad \tilde{y}=\tau-\frac{h}{12}(\xi-2\eta), \tag{4.5}$$

is the maximizer of the problem (4.3). This is possible if and only if $\xi \leqslant 0$, $\eta \geqslant 0$, and the optimal value $H_i^*(\xi, \eta)=d(\tilde{x}, \tilde{y})$ is as given in (4.4).

In the second case, the maximizer is located on the line $2x+y=3\tau$, $x \leqslant \tau$. This is valid only if $\xi \geqslant 0$. The maximizer is the solution of

$$d(x, y)=c, \qquad 2x+y=3\tau,$$

where the constant c is to be chosen such that exactly one solution exists. One now gets

$$\bar{x}=\tau+\frac{h}{24}(\xi-2\eta), \qquad \bar{y}=\tau-\frac{h}{12}(\xi-2\eta)$$

whenever $\bar{x} \leqslant \tau$; that is, $\xi-2\eta \leqslant 0$. And $H_i^*(\xi, \eta)=d(\bar{x}, \bar{y})$ with $H_i^*(\xi, \eta)$ from (4.4) can be confirmed. The third case is analogous to the second. The maximizer is now lying on $x+2y=3\tau$, $y \geqslant \tau$.

The last case occurs for $\xi-2\eta \geqslant 0$, $2\xi-\eta \geqslant 0$. Then the maximizer is in the vertex of the cone K_i, $\bar{x}=\bar{y}=\tau$, and $H_i^*(\xi, \eta)=d(\bar{x}, \bar{y})=\tau_i(\xi+\eta)$. Thus the proof is complete.

Using Proposition 4 and (1.2) one is led to the following program DC which is dual to PC.

PROGRAM DC.

$$\text{Maximize} -\sum_{i=1}^{n} H_i^*(p_{i-1}, -p_i) \qquad \text{with} \quad p_0=p_n=0, \tag{4.6}$$

where H_i^* is given by (4.4).

Next, by means of Theorem 2 the solvability of DC is shown. For this, let the feasible domain K of PC not be empty. Obviously, for $(m_0, m_1, ..., m_n) \in K$ it follows that

$$z=(m_0, m_1, m_1, ..., m_{n-1}, m_{n-1}, m_n) \in \operatorname{dom} f \cap \operatorname{dom} g$$

if, with $W_i=K_i$, f and g are given by (3.8) and (3.9), respectively. Now, by stability statements obtainable from [8, Sections 4.8, 5.3, and 5.5] the functions are proven to be z-stable. Indeed, F_i considered as a function defined in R^{2n} is z-stable, since z is an interior point of dom F_i. Further, in view of

$$I(x, y; K_i)=I(x, y; K_{1i})+I(x, y; K_{2i})$$

with $K_{1i} = \{(x, y): 2x + y \leqslant 3\tau_i\}$, $K_{2i} = \{(x,y): x + 2y \geqslant 3\tau_i\}$ and, since $I(\cdot, K_{1i})$, $I(\cdot, K_{2i})$ in spite of the affine constraints are polyhedral, the function $I(\cdot, K_i)$ is z-stable. Thus, this property holds also for f. In the same manner g is shown to be z-stable.

Therefore, by the Theorem 2, program D possesses an optimal solution and thus likewise Program DC. Summarizing the main result of this section concerning the convex spline interpolation reads of follows.

THEOREM 5. *Both Programs PC and DC are solvable if and only if the feasible domain K of PC is nonempty, and then the optimal values are equal. Furthermore, if a solution* $(p_0, p_1, ..., p_n)$ *of DC is known, the solution* $(m_0, m_1, ..., m_n)$ *of PC is given by formula* (3.13) *where* H_i^* *is to be defined by* (4.4).

As mentioned in the Introduction this theorem can be found already in paper [1]. There it is derived in a more direct way be starting from the Kuhn–Tucker conditions. Numerical tests presented in [1] show the strategy of solving PC via DC to be very efficient even if DC is handled by the ordinary Newton method; see also [11], [14], and [16].

5. MONOTONE SPLINE INTERPOLATION

As stated before, the monotone spline interpolation leads to

PROGRAM PM.

$$\begin{aligned} &\text{Minimize } \sum_{i=1}^{n} F_i(m_{i-1}, m_i) \\ &\text{subject to } (m_{i-1}, m_i) \in E_i \cup D_i \qquad (i = 1, ..., n), \end{aligned} \tag{5.1}$$

where F_i, E_i, and D_i are given by (2.9), (2.6), and (2.7). Here the feasible domain is always nonempty. Therefore, in view of Proposition 1, PM is always uniquely solvable but, as in the convex case, the solution can be computed more effectively by means of the corresponding dual program DM. For formulating this program the subprogram (1.3), now

$$\begin{aligned} &\text{maximize } d(x, y) = \xi x + \eta y - \frac{4}{h}\{(x-\tau)^2 + (x-\tau)(y-\tau) + (y-\tau)^2\} \\ &\text{subject to } (x, y) \in E \cup D \end{aligned} \tag{5.2}$$

with $h = h_i$, $\tau = \tau_i$, $E = E_i$, and $D = D_i$, is to be considered for all $(\xi, \eta) \in R^2$.

PROPOSITION 6. *The optimal value* $d_{\max} = H_i^*(\xi, \eta)$ *of program* (5.2) *is, with* $\sigma_i = 12\tau_i/h_i$, *given by*

$$
\begin{aligned}
\frac{12}{h_i} H_i^*(\xi,\eta) &= \sigma_i(\xi+\eta) + \xi^2 - \xi\eta + \eta^2 && \text{for } (\xi,\eta) \in A_i \cup B_i \\
&= -\tfrac{1}{4}\sigma_i^2 + \tfrac{3}{2}\sigma_i\xi + \tfrac{3}{4}\xi^2 && \text{for } (\xi,\eta) \in S_{1i} \\
&= -\tfrac{1}{4}\sigma_i^2 + \tfrac{3}{2}\sigma_i\eta + \tfrac{3}{4}\eta^2 && \text{for } (\xi,\eta) \in S_{2i} \\
&= -\sigma_i^2 && \text{for } (\xi,\eta) \in T_i \\
&= -2\sigma_i^2 + 2\sigma_i(\xi+\eta) \\
&\quad + 2\sigma_i\sqrt{\xi^2 - \xi\eta + \eta^2 - \sigma_i(\xi+\eta) + \sigma_i^2} && \text{for } (\xi,\eta) \in C_i
\end{aligned}
\tag{5.3}
$$

where

$$
\begin{aligned}
A_i &= \{(\xi,\eta) \in R^2 : -\sigma_i \leqslant 2\xi - \eta \leqslant 2\sigma_i,\ -2\sigma_i \leqslant \xi - 2\eta \leqslant \sigma_i\}, \\
B_i &= \{(\xi,\eta) \in R^2 : \xi^2 - \xi\eta + \eta^2 - \sigma_i(\xi+\eta) \leqslant 0\}, \\
S_{1i} &= \{(\xi,\eta) \in R^2 : \xi - 2\eta \geqslant \sigma_i,\ -\sigma_i \leqslant \xi \leqslant \sigma_i\}, \\
S_{2i} &= \{(\xi,\eta) \in R^2 : 2\xi - \eta \leqslant -\sigma_i,\ -\sigma_i \leqslant \eta \leqslant \sigma_i\}, \\
T_i &= \{(\xi,\eta) \in R^2 : \xi \leqslant -\sigma_i,\ \eta \leqslant -\sigma_i\}, \\
C_i &= R^2 \setminus (A_i \cup B_i \cup S_{1i} \cup S_{2i} \cup T_i).
\end{aligned}
$$

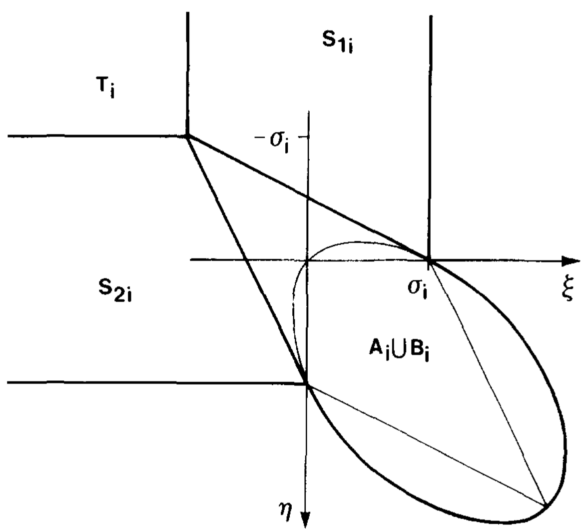

FIG. 4. Pieces of the domain of function (5.3).

The proof of (5.3) is elementary. Only a sketch will be given. Compare also the proof of (4.4). In the first case the maximizer is equal to the center (4.5) of the ellipse $d(x, y) = c$, valid if and only if $(\xi, \eta) \in A_i \cup B_i$. In the second case the maximizer $(\bar{x}, \bar{y})$ is lying on $y = 0$, $0 \leqslant x \leqslant 3\tau$. Then one gets

$$\bar{x} = \tfrac{3}{2}\tau + \tfrac{1}{8}h\xi, \qquad \bar{y} = 0$$

if $(\xi, \eta) \in S_{1i}$. The third case is symmetric to the second. In the fourth case the maximizer equals $\bar{x} = \bar{y} = 0$ true for $(\xi, \eta) \in T_i$. The last case occurs if the maximizer is located on the "outer" boundary of the ellipse

$$e(x, y) = \frac{4}{h}\{x^2 + xy + y^2 - 6\tau(x + y) + 9\tau^2\} = 0.$$

Then, the maximizer $(\bar{x}, \bar{y})$ solves

$$d(x, y) = c, \; e(x, y) = 0$$

if c is chosen to be the greatest value such that exactly one solution exists. Hence, by adding the two equations one gets

$$\alpha x + \beta y = \gamma$$

with $\alpha = \xi - 12\tau/h$, $\beta = \eta - 12\tau/h$, $\gamma = c - 24\tau^2/h$. Eliminating y in $e(x, y) = 0$, a quadratic equation in x arises. This has only one solution if

$$\gamma^2 - 4\tau(\alpha + \beta)\gamma + 12\tau^2\alpha\beta = 0.$$

The greater root is

$$\gamma = 2\tau\{\alpha + \beta + \sqrt{\alpha^2 - \alpha\beta + \beta^2}\}.$$

Now, because of $d_{\max} = d_{\max} + e_{\max} = \gamma + 24\tau^2/h$ the value of $H_i^*(\xi, \eta)$ given in (5.3) immediately results if $(\xi, \eta) \in C_i$. Thus the proof is complete.

By means of Proposition 6 and (1.2) the following program DM is obtained which is dual to PM.

PROGRAM DM.

$$\text{Maximize} - \sum_{i=1}^{n} H_i^*(p_{i-1}, -p_i) \qquad \text{with} \quad p_0 = p_n = 0, \tag{5.4}$$

where H_i^* is defined by (5.3).

In order to verify the solvability of DM suppose that, in sharpening (2.2),

$$\tau_1 > 0, ..., \tau_n > 0 \tag{5.5}$$

holds. Then, e.g., the vector

$$z = (m_0, M_1, m_1, ..., M_{n-1}, m_n, M_n) \in R^{2n}$$

with $m_i = M_i = \varepsilon/2$, $\varepsilon = \min\{\tau_1, ..., \tau_n\} > 0$ is an interior point of dom H_i even if the function H_i, defined by (3.7) with $W_i = E_i \cup D_i$, is considered as a function given on R^{2n}. Thus F_i is z-stable. In view of (3.8) one gets $z \in \operatorname{dom} f$ and the z-stability of f. Further, for the function g defined by (3.9) obviously $z \in \operatorname{dom} g$. And g is z-stable because g can be represented by a sum of polyhedral functions.

Therefore, Theorem 2 assures the existence of an optimal solution for Program DM. Summarizing, the main result of this paper concerning the monotone spline interpolation is the following.

THEOREM 7. *Both programs PM and DM possess optimal solutions if* (5.5) *holds, and then the optimal values are equal. Moreover, if a solution* $(p_0, p_1, ..., p_n)$ *of DM is given, the solution* $(m_0, m_1, ..., m_n)$ *of PM can be computed by means of formula* (3.13), *where* H_i^* *is to be defined by* (5.3).

6. PROGRAMS WITH GIVEN BOUNDARY VALUES

In the monotone spline interpolation

$$\tau_j = 0 \qquad \text{for some } j \in \{1, ..., n\} \tag{6.1}$$

may occur. Then, $m_{j-1} = m_j = 0$ for the solution of PM follows, and the optimization problem PM can be decomposed into subproblems of lower dimension. Analogously, if in the convex spline interpolation

$$\tau_j = \tau_{j+1} \qquad \text{for some } j \in \{1, ..., n-1\} \tag{6.2}$$

appears one gets $m_{j-1} = m_j = m_{j+1} = \tau_j$, and Program PC divides into lower dimensional subprograms.

Of course, from a numerical point of view such a decomposition should be undertaken if possible. In the resulting subproblems, one or both of the boundary values, say m_0 and m_n, are prescribed. For example, let $m_0 = \alpha$ and $m_n = \beta$ be given. Then one has to consider the slightly modified

PROGRAM PB.

$$\begin{aligned}&\text{Minimize } \sum_{i=1}^{n} F_i(m_{i-1}, m_i)\\ &\text{subject to } (m_{i-1}, m_i) \in W_i (i=1, ..., n), \qquad m_0=\alpha, m_n=\beta.\end{aligned} \tag{6.3}$$

Two ways are offered to derive a corresponding dual program. In the first case, define the functions f and g by (3.8) and (3.9) and let the set W now be

$$W = \{u \in R^{2n}: M_i = m_i (i=1, ..., n), m_0 = \alpha, m_n = \beta\}. \tag{6.4}$$

Then, it is immediately seen that

$$\begin{aligned} g^*(v) &= \alpha p_0 + \beta P_n \qquad \text{for} \quad p_i + P_i = 0 \; (i=1, ..., n) \\ &= +\infty \qquad \text{otherwise.} \end{aligned} \tag{6.5}$$

Using the functions H_i^* given by (1.3) the dual program now obtained reads as follows.

PROGRAM DB_1.

$$\text{Maximize } -\sum_{i=1}^{n} H_i^*(p_{i-1}, -p_i) + \alpha p_0 - \beta p_n. \tag{6.6}$$

In a second case, the variables m_0 and m_n are eliminated, and thus PB is considered to be a problem in the space R^{2n-2}. Following the line described in Section 3 one is led to the dual

PROGRAM DB_2.

$$\text{Maximize } -G_1^*(-p_1) - \sum_{i=2}^{n-1} H_i^*(p_{i-1}, -p_i) - G_n^*(p_{n-1}). \tag{6.7}$$

Here H_i^* are defined by (1.3) and G_1^*, G_n^* by

$$G_1^*(\eta) = \sup\{\eta y - F_1(\alpha, y): (\alpha, y) \in W_1\}, \tag{6.8}$$

$$G_n^*(\xi) = \sup\{\xi x - F_n(x, \beta): (x, \beta) \in W_n\}. \tag{6.9}$$

In the shape preserving spline interpolation, if (6.1) or (6.2) occur, it is recommended to decompose the program for determining the interpolant with minimal curvature into lower dimensional subprograms. These are programs with boundary conditions of the type PB, to be solved via DB_1 or DB_2. For the convex spline problem, Theorem 5 holds verbatim if PB is coupled with DB_1 or with DB_2, while for the monotone spline problem, an analog to Theorem 7 can be assured if PB is coupled with DB_2. The details are omitted because their verification is now straightforward.

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